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Noncommutative waves have infinite propagation speed

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Abstract: We prove the existence of global solutions to the Cauchy problem for noncommutative nonlinear wave equations in arbitrary even spatial dimensions where the noncommutativity is only in the spatial directions. We find that for existence there are no conditions on the degree of the nonlinearity provided the potential is positive. We furthermore prove that nonlinear noncommutative waves have infinite propagation speed, i.e., if the initial conditions at time 0 have a compact support then for any positive time the support of the solution can be arbitrarily large.

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Contents

1. Introduction

Classical solutions to the static equations of field theories in noncommutative spaces have been studied in some detail in recent years. Most of these theories are defined in the Moyal plane, its higher dimensional generalizations or on the fuzzy sphere. Explicit solitonic solutions have been found in various gauge theories, see, e.g., [1]–[5] as well as in scalar field theories [6]–[8] at infinite noncommutativity where the existence theory for finite noncommutativity is now rather complete [9]–[12] especially for the rotationally invariant case. For general background and reviews of noncommutative field theory we refer to [13]–[15].

Some time dependent solutions to noncommutative field equations have been found in gauge theories and sigma models, see, e.g., [16]–[22], and also by boosting static solutions in scalar field theories [23]. The main purpose of this paper is to study the initial value problem for nonlinear wave equations in odd dimensional noncommutative lorentzian spaces with the noncommutativity in the spatial directions only. We prove the existence of global solutions for a large class of initial conditions provided the nonlinear term in the wave equation is the derivative of a positive polynomial which vanishes to second order at zero. We do not need to impose any conditions on the degree of the nonlinearity. This is in stark contrast with the classical case where the existence theory depends strongly on the dimension of space as well as the nonlinearity [24].

Our existence results are obtained by adapting the standard theory of evolution equations in Banach spaces, see, e.g., [25, 26] and references therein, to the setting of noncommutative wave equations. The arguments presented in this paper can be generalized to nonpolynomial nonlinearities, e.g., entire functions.

We analyse the support properties of the solutions and show quite generally that the support is arbitrarily large in all spatial slices at positive times even though the initial data at time 0 have compact supports. Again this is qualitatively different from classical nonlinear waves which always have a propagation speed equal to that of the corresponding linear equation [24]. On the other hand, it should be noted that the noncommutative wave equation under consideration is not Lorentz invariant, since a specific choice of spatial slicing of Minkowski space is assumed.

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2. Existence of solutions

Let ∇^2 denote the laplacian in $\mathbb{R}^{2d}$. For simplicity we choose the nonlinear term in the wave equation to be a polynomial F. We assume $F(s) = V'(s)$ where $V(s) > 0$ for all $s \neq 0$, $V(0) = V'(0) = 0$ but $V''(0) > 0$. We study wave equations of the form

$$(\partial_t^2 - \nabla^2)\varphi(t,x) + F_*(\varphi)(t,x) = 0\,, \tag{2.1}$$

where $F_*(\varphi)$ is defined by replacing the ordinary pointwise product of functions by the standard star product for functions on $\mathbb{R}^{2d}$, i.e.,

$$F_*(\varphi) = \sum_{n=1}^{N} c_n\, \varphi^{*n}\,, \tag{2.2}$$

if $F(s) = \sum_{n=1}^{N} c_n\, s^n$ and φ^{*n} denotes the n'th $*$-power of φ. The star product of two functions f and g is defined by

$$(f * g)(t,x) = \exp\left(-\frac{i}{2}\Theta_{ij}\frac{\partial}{\partial y_i}\frac{\partial}{\partial z_j}\right) f(t,y)g(t,z)\Big|_{y=z=x}\,, \tag{2.3}$$

where Θ is a nondegenerate antisymmetric $2d \times 2d$ matrix. A more enlightening formula for the star product, which clearly exhibits its nonlocal character, is given by

$$(f * g)(t,x) = \frac{1}{(2\pi)^{2d}}\int f(t, x + \frac{1}{2}\Theta p)g(t, x+y)e^{ip\cdot y}\, dpdy \tag{2.4}$$

which can be obtained from (2.3) by Fourier transforming.

We are interested in real valued solutions φ to eq. (2.1). We will assume for simplicity that the noncommutativity matrix Θ is a direct sum of d 2×2 antisymmetric matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

times a positive noncommutativity parameter, denoted θ. Our results are, however, valid for arbitrary non-degenerate antisymmetric Θ.

It is convenient to apply the Weyl quantization map to the wave equation and then, with our conventions, it takes the form

$$\partial_t^2\phi(t) + 2\theta^{-1}\sum_{k=1}^{d}[a_k^*, [a_k, \phi(t)]] + F(\phi(t)) = 0\,, \tag{2.5}$$

where $\phi(t)$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$ and the operators a_k^* and a_k are d independent pairs of raising and lowering operators whose explicit form can be taken to be

$$a_k = \frac{1}{\sqrt{2}}\left(\xi_k + \frac{\partial}{\partial \xi_k}\right), \tag{2.6}$$

where ξ_k, $k = 1, \ldots, d$, are the coordinates in $\mathbb{R}^d$. The relation between a function φ on $\mathbb{R}^{2d}$ and the corresponding Weyl operator $\pi(\varphi) = \phi$ on $L^2(\mathbb{R}^d)$ is given by

$$\phi = \frac{1}{(2\pi)^d} \int \tilde{\varphi}(w_1, \ldots, w_{2d}) \exp\left(i \sum_{j=1}^{d} (w_{2j-1}\hat{\xi}_j + w_{2j}\hat{p}_j) \right) dw\,, \tag{2.7}$$

where $\hat{\xi}_k$ and $\hat{p}_k$ are the position and momentum operators corresponding to the raising and lowering operators a_k^* and a_k and $\tilde{\varphi}$ denotes the Fourier transform of φ. The Weyl map is in fact an algebra isomorphism from $L^2(\mathbb{R}^{2d})$ (with the star product) to the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$ and takes real valued functions to self-adjoint operators.

The collection of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$ forms a Hilbert space, denoted $\mathcal{H}^2$, with inner product

$$\langle A, B \rangle_2 = \mathrm{Tr}(A^* B)\,. \tag{2.8}$$

We define a self-adjoint operator Δ on $\mathcal{H}^2$ by

$$\Delta \pi(\varphi) = \pi(\nabla^2 \varphi) \tag{2.9}$$

and it is easily checked that

$$\Delta \phi = -2 \sum_{k=1}^{d} [a_k^*, [a_k, \phi]]\,. \tag{2.10}$$

Its domain is $\mathcal{D}(\Delta) = \{\pi(\varphi) :\ \varphi, \nabla^2\varphi \in L^2(\mathbb{R}^d)\}$.

It is convenient to write the wave equation as two first order equations in t:

$$\partial_t \phi_1 = \phi_2 \tag{2.11}$$

$$\partial_t \phi_2 = \Delta \phi_1 - \theta F(\phi_1)\,. \tag{2.12}$$

Here we have also rescaled the time variable by $\theta^{-1/2}$. Writing

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{2.13}$$

for ordered pairs of operators, and defining

$$A = i \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \qquad J \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -F(\phi_1) \end{pmatrix}, \tag{2.14}$$

we can write the wave equation on the form

$$\partial_t \Phi = -iA\Phi + \theta J(\Phi)\,, \tag{2.15}$$

where A is a linear operator acting on pairs of operators.

We now prove existence of solutions to (2.15) using standard mehods from the theory of nonlinear evolution equations which can also be used to deal with ordinary nonlinear wave equations, see [25, 26]. First let us note that we can replace the operator Δ in (2.15) by $\Delta - m^2$, $m > 0$, by adding a compensating linear term to F and the modified F is still the derivative of a positive polynomial, provided m is small enough. Define a self-adjoint

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operator B on $\mathcal{H}^2$ by

$$B = (-\Delta + m^2)^{1/2} \,. \tag{2.16}$$

Then $B \geq m$ and we define the space $\mathcal{H}^{1,2}$ as the set of operators $\phi \in \mathcal{H}^2$ for which $B\phi \in \mathcal{H}^2$. Let $\mathcal{H} = \mathcal{H}^{1,2} \oplus \mathcal{H}^2$ and define an inner product $\langle \cdot \,, \cdot \rangle$ on $\mathcal{H}$ by

$$\left\langle \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\rangle = \langle B\phi_1, B\psi_1 \rangle_2 + \langle \phi_2, \psi_2 \rangle_2 \,. \tag{2.17}$$

With this inner product $\mathcal{H}$ is a Hilbert space and the operator

$$D = i \begin{pmatrix} 0 & I \\ -B^2 & 0 \end{pmatrix} \tag{2.18}$$

is a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(D) = \mathcal{D}(\Delta) \oplus \mathcal{H}^2$. We can now apply the following theorem.

Theorem. *Let T be a self-adjoint operator with domain $\mathcal{D}(T)$ on a Hilbert space H with norm $\| \cdot \|$ and $N : \mathcal{D}(T) \mapsto \mathcal{D}(T)$ a mapping such that the following conditions hold for any $\phi, \psi \in \mathcal{D}(T)$:*

(i) $\|N(\phi)\| \leq C_1(\|\phi\|)\|\phi\|$

(ii) $\|TN(\phi)\| \leq C_2(\|\phi\|)\|T\phi\|$

(iii) $\|N(\phi) - N(\psi)\| \leq C_3(\|\phi\|, \|\psi\|)\|\phi - \psi\|$

(iv) $\|T(N(\phi) - N(\psi))\| \leq C_4(\|\phi\|, \|\psi\|, \|T\phi\|, \|T\psi\|)\|T\phi - T\psi\|$

where the constants C_j, $j = 1, 2, 3, 4$, are increasing functions of the norms. Then, for any $\phi_0 \in \mathcal{D}(T)$ there exists $t_0 > 0$ and a unique continuously differentiable family of vectors $\{\phi(t)\}_{0 \leq t < t_0} \subseteq \mathcal{D}(T)$ such that

$$\partial_t \phi(t) = -iT\phi(t) + N(\phi(t)) \tag{2.19}$$

and

$$\phi(0) = \phi_0 \,. \tag{2.20}$$

If $\|\phi(t)\|$ is bounded by a constant independent of t then one can take $t_0 = \infty$.

A proof of this theorem can be found in [26]. The basic step is to convert the differential equation (together with the initial conditions) into the integral equation

$$\phi(t) = e^{-itT}\phi_0 + \int_0^t e^{-i(t-s)T} N(\phi(t)) \, ds \,. \tag{2.21}$$

It is quite straightforward to verify conditions (i)-(iv) for $N = \theta J$, $T = D$ and the Hilbert space $\mathcal{H}$ that we defined before. The uniform bound on the norm of the solution is obtained from the conservation of energy.

For illustration we verify (i) and (iv) leaving the others, which are similar, to the reader. Let us denote the norm on the space of Hilbert-Schmidt operators by $\|\cdot\|_2$. Then we have for the norm $\|\cdot\|$ on $\mathcal{H}$

$$\left\|\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right\|^2 = \|B\phi_1\|_2^2 + \|\phi_2\|_2^2 \tag{2.22}$$

and clearly

$$\left\|J\left(\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right)\right\|^2 = \|F(\phi_1)\|_2^2\,. \tag{2.23}$$

For (i) we therefore need to show that

$$\|F(\phi_1)\|_2 \leq C_1(\|B\phi_1\|_2)\|B\phi_1\|_2\,. \tag{2.24}$$

Since F is a polynomial it suffices to verify the inequality (2.24) for the case $F(s) = s^n$ and this follows from

$$\begin{aligned}\|\phi_1^n\|_2 &\leq \|\phi_1\|_2^n\\ &\leq \left(m^{-n}\|B\phi_1\|_2^{n-1}\right)\|B\phi_1\|_2.\end{aligned} \tag{2.25}$$

For condition (iv) it is again enough to consider the case when F is a monomial. With F as before we have

$$\left\|DJ\left(\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right) - DJ\left(\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right)\right\| = \|B\phi_1^n - B\psi_1^n\|_2 \tag{2.26}$$

and

$$\left\|D\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} - D\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right\|^2 = \|B(\phi_2-\psi_2)\|_2^2 + \|B^2(\phi_1-\psi_1)\|_2^2 \tag{2.27}$$

so we need to show that

$$\|B(\phi_1^n - \psi_1^n)\|_2 \leq C\,\|B^2(\phi_1-\psi_1)\|_2\,, \tag{2.28}$$

where the constant C is an increasing function of the norms as stated in the theorem. Let us define operators A_k on $\mathcal{H}^{1,2}$ by

$$A_k\phi = [a_k, \phi]\,, \tag{2.29}$$

for $k = 1,\ldots,d$. The operators A_k are derivations and

$$-B^2 + m^2 = \Delta = -2\sum_{k=1}^{d} A_k^* A_k\,. \tag{2.30}$$

We can write

$$\phi_1^n - \psi_1^n = \sum_{i=0}^{n-1}\phi_1^{n-1-i}(\phi_1-\psi_1)\psi_1^i \tag{2.31}$$

so in order to establish (2.28) it suffices to estimate

$$\|B(\phi_1^i(\phi_1-\psi_1)\psi_1^j)\|_2^2 = m^2\|\phi_1^i(\phi_1-\psi_1)\psi_1^j)\|_2^2 + 2\sum_{k=1}^{d}\|A_k(\phi_1^i(\phi_1-\psi_1)\psi_1^j)\|_2^2 \tag{2.32}$$

with $i+j=n-1$. The first term on the right hand side above is bounded by

$$m^{-2}\|\phi_1\|_2^{2i}\|\psi_1\|_2^{2j}\|B^2(\phi_1-\psi_1)\|_2^2$$

so it suffices to consider the second term. We have

$$\begin{aligned}\|A_k(\phi_1^i(\phi_1-\psi_1)\psi_1^j)\|_2 \leq\ & i\,\|A_k\phi_1\|\,\|\phi_1\|^{i-1}\|\phi_1-\psi_1\|_2\|\psi_1\|^j+\|\phi_1\|^i\|A_k(\phi_1-\psi_1)\|_2\|\psi_1\|^j+\\ &+j\,\|\phi_1\|^i\|\phi_1-\psi_1\|_2\|\,\|A_k\psi_1\|\,\|\psi_1\|^{j-1}.\end{aligned} \tag{2.33}$$

Since

$$\|A_k\chi\|_2 \leq \|B\chi\|_2 \tag{2.34}$$

for $k=1,\ldots,d$ and any χ in the domain of B the desired result follows easily by summing over k.

It follows from the differentiability of the solution

$$\Phi(t) = \begin{pmatrix} \phi_1(t) \\ \partial_t\phi_1(t) \end{pmatrix} \tag{2.35}$$

and the equation of motion that the energy

$$E(t) = \mathrm{Tr}\left(\sum_{k=1}^{d}[a_k,\phi_1(t)][\phi_1(t),a_k^*] + \frac{1}{2}(\partial_t\phi_1(t))^2 + \theta V(\phi_1(t))\right) \tag{2.36}$$

is independent of t. Since $V(x) \geq \frac{1}{2}m^2x^2$ for real x it follows that, for $\varepsilon > 0$ sufficiently small, we have

$$E(t) \geq \varepsilon\|\Phi(t)\|^2 \tag{2.37}$$

provided $\Phi(t)$ is self-adjoint, which is the case if the initial conditions are self-adjoint. Thus, we have proven that *all solutions to eq. (2.1) with real initial data* $\varphi(0,\cdot)\in\mathcal{D}(\nabla^2)$ *and* $\partial_t\varphi(0,\cdot)\in L^2(\mathbb{R}^{2d})$ *are global.*

We remark that the estimates above for the noncommutative case are in general not valid for the corresponding classical (commutative) nonlinear equations. For example, the Sobolev inequality needed to establish the conditions (i)–(iv) in the classical case is valid for arbitrary polynomials F if $d=1$, whereas for $d\geq 2$ it only holds for linear F. Star powers of functions are more regular than ordinary powers of functions. Roughly speaking, the local singularities of the star product of two functions are no worse than the singularities of the individual functions. We will discuss this in more detail in the following section. It follows from this enhanced regularity that the existence problem for nonommutative nonlinear waves is mathematically similar to the existence problem for nonlinear waves in the case where space has been replaced by a discrete lattice.

3. Infinite propagation speed

In this section we show that in general the diameter of the support of a solution to the noncommutative wave equation increases in time with infinite speed. By support we here mean the support of the function which corresponds to the operator under the Weyl map.

We first note that the solutions whose existence was established in the preceding section can be written

$$\phi(t) = (\cos tB)\phi(0) + (B^{-1}\sin tB)\partial_t\phi(0) - \theta\int_0^t B^{-1}\sin((t-s)B)F(\phi(s))\,ds \tag{3.1}$$

by using (2.21) and the expression

$$e^{-itD} = \begin{pmatrix} \cos tB & B^{-1}\sin tB \\ -B\sin tB & \cos tB \end{pmatrix} \tag{3.2}$$

for the linear time development operator acting on $\mathcal{H}$. Let $\varphi(t,\cdot)$ be the function corresponding to $\phi(t)$ under the Weyl map, let $g(t,\cdot)$ correspond to

$$\chi(t) = (\cos tB)\phi(0) + (B^{-1}\sin tB)\partial_t\phi(0) \tag{3.3}$$

and let $h(t,\cdot)$ correspond to

$$\psi(t) = \theta\int_0^t B^{-1}\sin((t-s)B)F(\phi(s))\,ds\,. \tag{3.4}$$

Then g is a solution to the linear wave equation and the support of g is contained in the ordinary causal future of the union of the supports of $\varphi(0,\cdot)$ and $\partial_t\varphi(0,\cdot)$ by the Huygens principle for solutions to linear wave equations. We are therefore interested in studying the support properties of $h(t,\cdot)$.

We note that $\phi(t)\to\phi_0$ and hence also $F(\phi(t))\to F(\phi_0)$ as $t\to 0$ where the convergence is in Hilbert-Schmidt norm. Since the operator $(tB)^{-1}\sin(tB)$ converges strongly to the identity operator I as $t\to 0$ and is uniformly bounded in norm it follows from eq. (3.4) that

$$\left\|\psi(t) - \frac{1}{2}t^2\theta F(\phi_0)\right\|_2 = o(t^2) \tag{3.5}$$

as $t\to 0$. It follows that the relevant support properties of $h(t,\cdot)$ for small t are determined by those of $F_*(\varphi_0)$ where $\phi_0 = \pi(\varphi_0)$. Suppose that φ_0 is smooth with a compact support S_0 and assume that we can show that the support of $F_*(\varphi_0)$ is strictly larger than S_0 in the sense that there is a closed ball U disjoint from S_0 with

$$\int_U |F_*(\varphi_0)|^2\,dx \equiv A > 0\,. \tag{3.6}$$

Then, by eq. (3.5), we have

$$\int_U |h(t,x)|^2\,dx \geq \frac{1}{2}t^2\theta A + o(t^2) \tag{3.7}$$

and $h(t,\cdot)$ is nonzero on a subset of U of positive measure for all t small enough. Evidently this proves that the propagation speed is infinite for the initial values $\varphi(0,\cdot) = \varphi_0$ and $\partial_t\varphi(0,\cdot)$ smooth with support in S_0.

It remains to demonstrate that the support of $F_*(\varphi_0)$ is in general strictly larger than that of φ_0. The support properties of star products of functions are simply reflected in the algebra of star products of δ-functions and their Fourier transforms. We define the distributions $D(\alpha)$, $E(\beta)$ for $\alpha, \beta \in \mathbb{R}^{2d}$ by

$$D(\alpha) = \pi^d \delta_\alpha \qquad \text{and} \qquad E(\beta) = e^{-2i\Theta(\beta,\cdot)}\,, \tag{3.8}$$

where δ_α is the delta function supported at α and Θ is the skew symmetric quadratic form on $\mathbb{R}^{2d}$ entering Eq. (2.3)

$$\Theta(\alpha,\beta) = \Theta_{ij}\alpha_i\beta_j\,. \tag{3.9}$$

By a straightforward computation, using (2.4), one finds

$$D(\alpha) * D(\beta) = e^{2i\Theta(\alpha,\beta)}E(\alpha-\beta) \tag{3.10}$$
$$D(\alpha) * E(\beta) = e^{2i\Theta(\alpha,\beta)}D(\alpha-\beta) \tag{3.11}$$
$$E(\alpha) * D(\beta) = e^{-2i\Theta(\alpha,\beta)}D(\alpha+\beta) \tag{3.12}$$
$$E(\alpha) * E(\beta) = e^{-2i\Theta(\alpha,\beta)}E(\alpha+\beta)\,. \tag{3.13}$$

The Weyl map can in fact be extended to distributions using (2.4) and one can show that the star product of two tempered distributions is again a tempered distribution provided they are sufficiently regular or one of the two distributions has a compact support.

We can write

$$f = \int f(\alpha)\delta_\alpha\, d\alpha \tag{3.14}$$

for a function f on $\mathbb{R}^d$ so

$$\begin{aligned}(f*f)(x) &= \pi^{-2d}\int f(\alpha)f(\beta)(D(\alpha)*D(\beta))(x)\,d\alpha d\beta \\ &= \pi^{-2d}\int f(\alpha)f(\beta)e^{2i\Theta(\alpha,\beta)}e^{2i\Theta(\alpha-\beta,x)}\,d\alpha d\beta \\ &= \pi^{-2d}\int f(\alpha+\beta)f(\beta)e^{2i\Theta(\alpha,\beta)}\,d\beta\, e^{2i\Theta(\alpha,x)}\,d\alpha\,.\end{aligned} \tag{3.15}$$

If f is, say, continuous with compact support, it follows from the last expression that $f*f$ is the Fourier transform of a function with compact support and therefore an entire analytic function. In particular, its support is $\mathbb{R}^{2d}$ unless $f=0$. A similar argument shows that all even star powers of f are analytic if f has a compact support.

Let us now discuss odd star powers and concentrate on the third star power for simplicity. We find as before that

$$\begin{aligned}(f*f*f)(x) &= \pi^{-3d}\int f(\alpha)f(\beta)f(\gamma)(D(\alpha)*D(\beta)*D(\gamma))(x)\,d\alpha d\beta d\gamma \qquad (3.16)\\ &= \pi^{-3d}\int f(\alpha)f(\beta)f(\gamma)e^{2i\Theta(\beta,\gamma)}e^{2i\Theta(\alpha,\beta-\gamma)}D(x-\alpha+\beta-\gamma)\,d\alpha d\beta d\gamma\,,\end{aligned}$$

from which it follows that the support of $f * f * f$ is contained in

$$\{\alpha + \beta - \gamma : \alpha, \beta, \gamma \in \operatorname{supp} f\} . \tag{3.17}$$

In particular, if the support of f has diameter R, the support of $f * f * f$ has diameter

$$R' \leq 3R . \tag{3.18}$$

In the case of, say, a quartic potential with a nonzero cubic term, i.e.,

$$F(s) = V'(s) = as^3 + bs^2 + cs , \tag{3.19}$$

where $b \neq 0$, we have therefore proven infinite propagation speed.

For generic functions f we have an equality in (3.18). Here, we will not elaborate on the detailed conditions under which this is valid. It suffices to consider an example which demonstrates infinite propagation speed in the case $b = 0$ in eq. (3.19). Let

$$f = D(\alpha) + D(-\alpha) , \tag{3.20}$$

where $\alpha \in \mathbb{R}^{2d}$ has euclidean norm $|\alpha| = R/2$ such that the support of f has diameter R. It is easy to see that

$$f * f * f = 3(D(\alpha) + D(-\alpha)) + D(3\alpha) + D(-3\alpha) \tag{3.21}$$

which has support diameter $R' = 3R$. Similarly, f raised to the $(2n+1)$st star power will contain delta functions at $\pm(2n+1)\alpha$. In order to get an example where f is a smooth function, let $d_n^{\pm}$ be two sequences of nonnegative smooth functions on $\mathbb{R}^{2d}$ with $d_n^{\pm}$ supported in a ball of radius $1/n$ around $\pm\alpha$ and such that

$$\int d_n^{\pm}(x)\, dx = 1 . \tag{3.22}$$

This ensures that

$$f_n \equiv \pi^d (d_n^+ + d_n^-) \tag{3.23}$$

is smooth and has a support diameter $R_n \leq R + 2/n$. Furthermore,

$$f_n \to f = D(\alpha) + D(-\alpha) \tag{3.24}$$

in the sense of distributions. It is easily seen from (3.16) that

$$f_n * f_n * f_n \to f * f * f \tag{3.25}$$

and hence the support diameter of $f_n * f_n * f_n$ converges to $3R$ as $n \to \infty$. A similar argument allows one to determine the support of arbitrary odd star powers of compactly supported functions.

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4. Discussion

We have proved the existence of global solutions to the initial value problem for noncommutative nonlinear wave equations whose non-linear term is the dervative of a positive interaction potential and with noncommutativity in the spatial directions only. The existence theory is quite simple and independent of the nonlinearity and the (even) dimension of space.

We have shown that the speed of propagation is infinite, meaning that the support of the solution is arbitrarily large at positive times given that the support of the function and its first time derivative are compact at time 0. The part of the solution which travels at infinite speed is proportional to the noncommutativity parameter θ.

We have discussed only real solutions, but the arguments can be extended to cover complex wave equations such as

$$\partial_t^2 \phi(t) + 2\theta^{-1} \sum_{k=1}^{d} [a_k^*, [a_k, \phi(t)]] + \phi(t) F(|\phi(t)|^2) = 0\,, \tag{4.1}$$

in operator form, where $|\phi(t)|^2 = \phi(t)^* \phi(t)$ and the function F is the derivative of a polynomial $V(s)$ that is positive for $s > 0$ and vanishes linearly at $s = 0$.

We can use the formalism set up in this paper to study scattering of noncommutative waves and this will be the subject of a forthcoming publication. It will be interesting to see whether this scattering theory can be used to analyse the scattering of noncommutative solitons and whether the results on soliton scattering at large θ obtained in [7, 27, 28] can be proven rigorously.

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